11-7 Binomial Expansions & Powers of Binomials P. 537

Consider the following expanded powers of $(a + b)^n$ where a + b is any binomial. Let's look for patterns:

$(a+b)^0 = 1$

$(a+b)^1 = 1a^1 + 1b^1$

$(a+b)^2 = 1a^2 + 2a^1b^1 + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b^1 + 3a^1b^2 + 1b^3$

$(a+b)^4 = 1a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + 1b^4$

$(a+b)^5 = 1a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1b^5$

The coefficients (in red) come from Pascal's Triangle, below:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Let's make some observations about the patterns above:

1) Each row begins and ends with a 1.

2) The second number in each row is the same as the power of the binomial before it is expanded.

3) The numbers in any row include the sums of the numbers from the row above (excluding the beginning and ending ones).

4) The powers of "a" are descending, while the powers of "b" are ascending.

5) The degree of every term is the same as the power of the binomial.

6) The expanded binomial has one more term than the power of the binomial.

7) The coefficients for any row are symmetric.

Ex 1) Expand the binomial $(a + b)^6$

$1a^6 + 6a^5b^1 + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + 1b^6$

Ex 2) Find the first four terms in the expansion of $(x - 2y)^7$ (We will find all the terms of the expansion for now)

$$1a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + 1b^7$$

Next, replace "a" with "x" and replace "b" with "-2y"

$$x^7 + 7x^6(-2y) + 21x^5(-2y)^2 + 35x^4(-2y)^3 + 35x^3(-2y)^4$$
$$+ 21x^2(-2y)^5 + 7x(-2y)^6 + (-2y)^7$$

Simplify exponents before doing any multiplication:

$$x^7 + 7x^6(-2y) + 21x^5(4y^2) + 35x^4(-8y^3) + 35x^3(16y^4)$$
$$+ 21x^2(-32y^5) + 7x(64y^6) + (-128y^7)$$

Multiply:

$$x^7 - 14x^6y + 84x^5y^2 - 280x^4y^3 + 560x^3y^4$$
$$- 672x^2y^5 + 448xy^6 - 128y^7$$

$$x^7 - 14 \cdot x^6 \cdot y + 84 \cdot x^5 \cdot y^2 - 280 \cdot x^4 \cdot y^3 + 560 \cdot x^3 \cdot y^4 - 672 \cdot x^2 \cdot y^5 + 448 \cdot x \cdot y^6 - 128 \cdot y^7$$

The complete expansion follows:

$$a^{20} + 20 \cdot a^{19} \cdot b + 190 \cdot a^{18} \cdot b^{2} + 1140 \cdot a^{17} \cdot b^{3} + 4845 \cdot a^{16} \cdot b^{4} + 15504 \cdot a^{15} \cdot b^{5} +$$

$$38760 \cdot a^{14} \cdot b^{6} + 77520 \cdot a^{13} \cdot b^{7} + 125970 \cdot a^{12} \cdot b^{8} + 167960 \cdot a^{11} \cdot b^{9} + 184756 \cdot a^{10} \cdot b^{10} +$$

$$167960 \cdot a^{9} \cdot b^{11} + 125970 \cdot a^{8} \cdot b^{12} + 77520 \cdot a^{7} \cdot b^{13} + 38760 \cdot a^{6} \cdot b^{14} + 15504 \cdot a^{5} \cdot b^{15} +$$

$$4845 \cdot a^{4} \cdot b^{16} + 1140 \cdot a^{3} \cdot b^{17} + 190 \cdot a^{2} \cdot b^{18} + 20 \cdot a \cdot b^{19} + b^{20}$$