The Fundamental Theorem of Algebra (Carl Gauss)
For every polynomial of degree $n > 1$(with complex coefficients) there exists at least one linear factor.

Another Theorem by Carl Friedrich Gauss
Every polynomial of degree $n > 1$, (with complex coefficients) can be factored into exactly n linear factors.

Once we have these n linear factors, we can use the Zero Product Property to find the n roots or solutions of the polynomial.

Conjugate Root Theorem for Complex Roots
If a polynomial P(x) of degree greater than or equal to 1 (with real coefficients) has a complex number as a root $a + bi$, then its conjugate $a - bi$ is also a root.
In other words, complex roots occur in conjugate pairs.

Conjugate Root Theorem for Irrational Roots
If a polynomial P(x) of degree greater than or equal to 1 (with rational number coefficients) has an irrational root $a + b\sqrt{c}$, then its conjugate $a - b\sqrt{c}$ is also a root.
In other words, irrational roots occur in conjugate pairs.

Ex 1) Find a cubic equation with integer coefficients that has 2 and 3 - *i* has roots.

3+i is also a root according to the conjugate root theorem.

$x = 2$; $x = 3 - i$; $x = 3 + i$ roots

$(x-2)[(x-3)+i][(x-3)-i] = 0$ factors

$(x-2)((x-3)^2 - i^2) = 0$

$(x-2)[(x^2-6x+9) - i^2] = 0$ Remember: $i^2 = -1$

$(x-2)(x^2-6x+9+1) = 0$

$(x-2)(x^2-6x+10) = 0$

$x^3 - 6x^2 + 10x - 2x^2 + 12x - 20 = 0$

$x^3 - 8x^2 + 22x - 20 = 0$

EX2 Solve $x^4 - 12x - 5 = 0$ if $-1+2i$ is a root

$-1-2i$ is also a root (conjugate root theorem)

$x = -1-2i$; $x = -1+2i$ $\rightarrow$ roots

$[(x+1)+2i][(x+1)-2i]$ $\rightarrow$ factors

$(x+1)^2 - 4i^2$

$x^2+2x+1 - 4(-1)$ remember $i^2 = -1$

$x^2+2x+1+4$

x^2+2x+5

$$
\begin{array}{r}
x^2 - 2x - 1 \\
x^2+2x+5 \,\big)\, \overline{x^4 + 0x^3 + 0x^2 - 12x - 5} \\
+\left(-x^4 - 2x^3 - 5x^2\right) \\
\hline
-2x^3 - 5x^2 - 12x \\
+\left(+2x^3 + 4x^2 + 10x\right) \\
\hline
-x^2 - 2x - 5 \\
+\left(+x^2 + 2x + 5\right) \\
\hline
0
\end{array}
$$

Solve:

$$x^2 - 2x - 1 = 0$$

$a = 1 \quad b = -2 \quad c = -1$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 + 4}}{2}$$

$$x = \frac{2 \pm \sqrt{8}}{2}$$

$$x = \frac{2 \pm \sqrt{4}\sqrt{2}}{2}$$

$$x = \frac{2 \pm 2\sqrt{2}}{2}$$

$$x = \frac{\cancel{2}(1 \pm \sqrt{2})}{\cancel{2}}$$

The 4 roots:

$x = 1 + \sqrt{2}$; $x = 1 - \sqrt{2}$

$x = -1 + 2i$; $x = -1 - 2i$

Descartes' Rule of Signs

The number of positive real roots of a polynomial P(x) (with real coefficients) is either:

1) the same as the number of variations of signs of P(x), or
2) a multiple of 2 less than the number of sign changes.

Corollary to Descartes' Rule of Signs

The number of negative real roots of a polynomial P(x) (with real coefficients) is either:

1) the number of sign variations of P(-x), or

2) a multiple of 2 less than the number of sign changes.

ex 3 $P(x) = x^5 + x^4 - 3x^2 + 4x + 6$

2 sign changes

2 or 0 positive real roots.

corollary

$$P(-x) = (-x)^5 + (-x)^4 - 3(-x)^2 + 4(-x) + 6$$

$$P(-x) = -x^5 + x^4 - 3x^2 - 4x + 6$$

3 sign changes

3, 1 negative real roots

The polynomial $P(x) = x^5 + x^4 - 3x^2 + 4x + 6$ has 5 roots.

2, 0 positive real roots

3, 1 negative real roots

0, 2, 4 complex roots