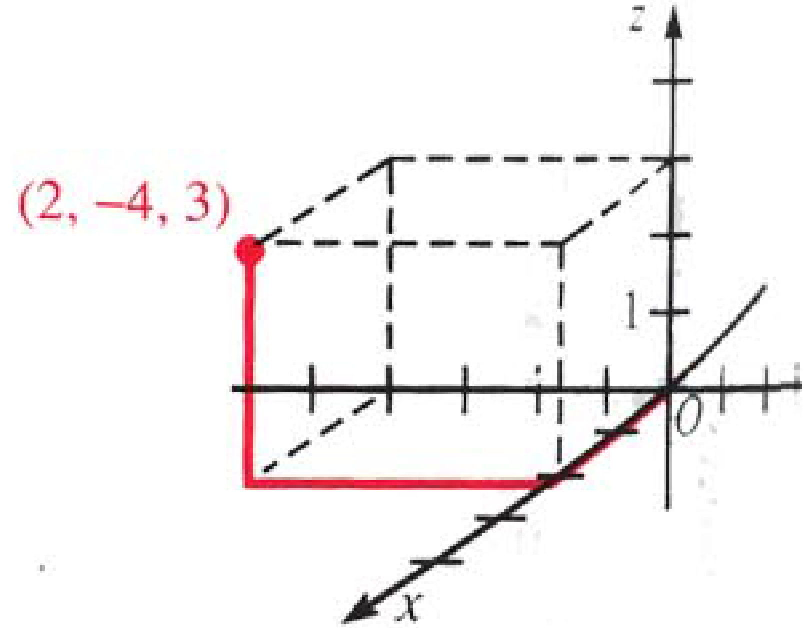
Alg 2 9-9 Systems of Linear Equations in 3 Variables Page 444

Alg. 2 Standard 2.0: Students solve systems of linear equations and inequalities (in two or three variables) by substitution, with graphs, or with matrices.

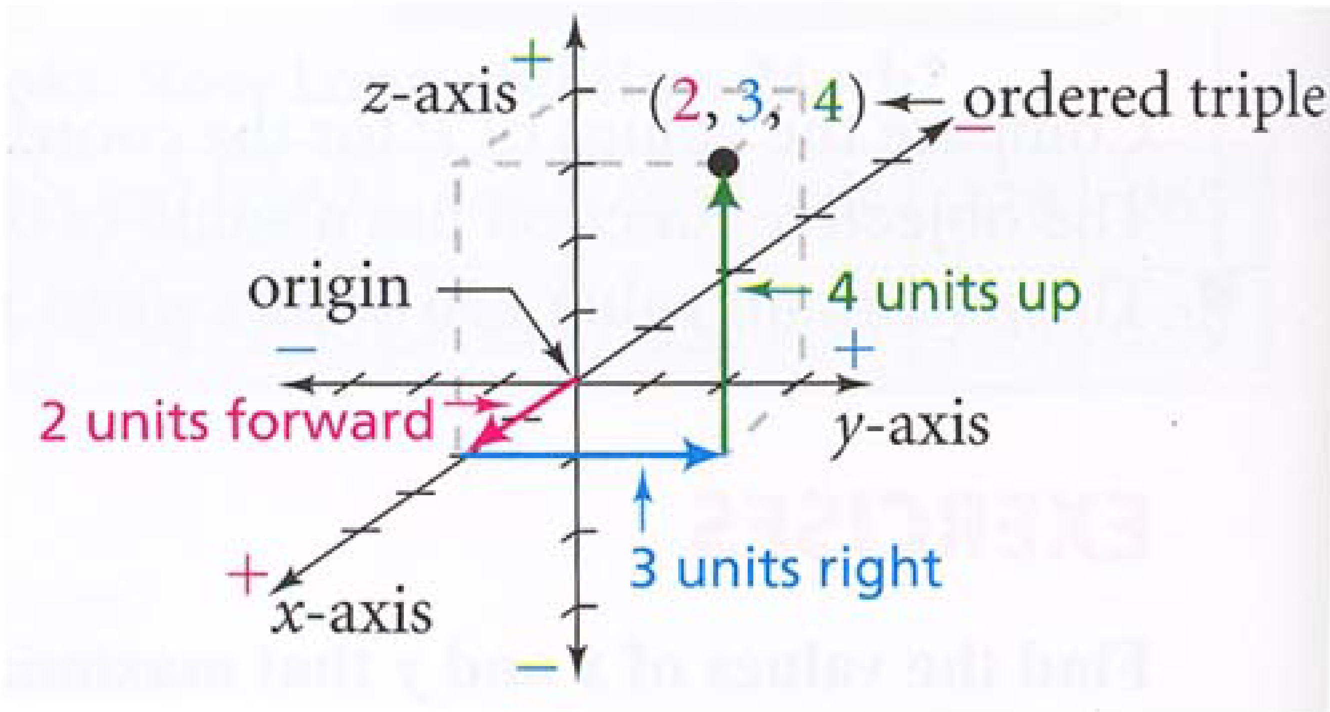
Objective: To solve systems of linear equations in three variables using Gaussian Elimination.

An **ordered triple** (x, y, z) is a solution to an equation that has three variables. For example, (2, -4, 3) is a solution to the equation $3x + 2y - z = -5$, because $3(2) + 2(-4) - 3 = -5$ is a true statement.

An ordered triple represents a point in a three dimensional space (coordinate space). The three dimensional coordinate system uses an x, y, and z axis. The point (2, -4, 3) can be represented using the graph below.



The point (2, 3, 4) can be represented using the graph below. Starting at the origin, we move 2 units forward, 3 units to the right, and 4 units up.

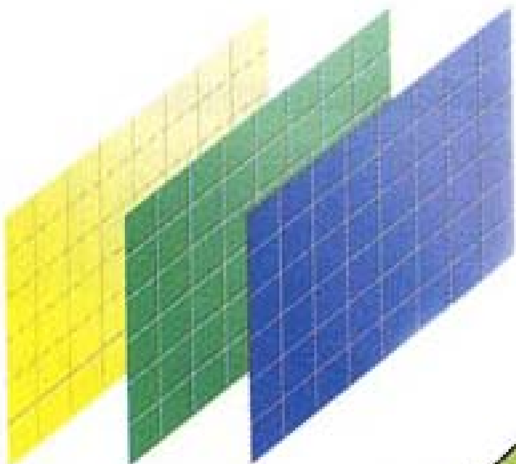


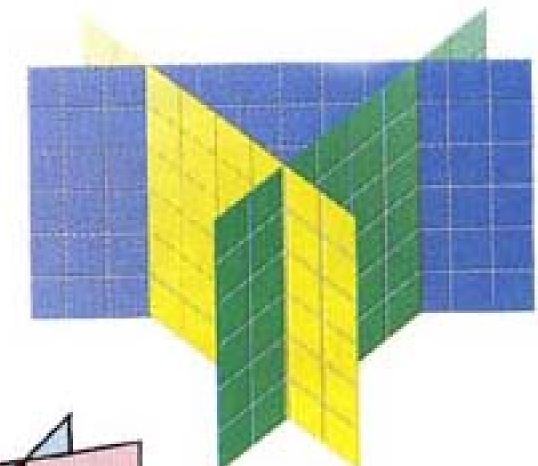
An equation in the form $Ax + By + Cz = D$, where A, B, C, and D are real numbers with A, B, and C not all zero is called a **linear equation in three variables**.

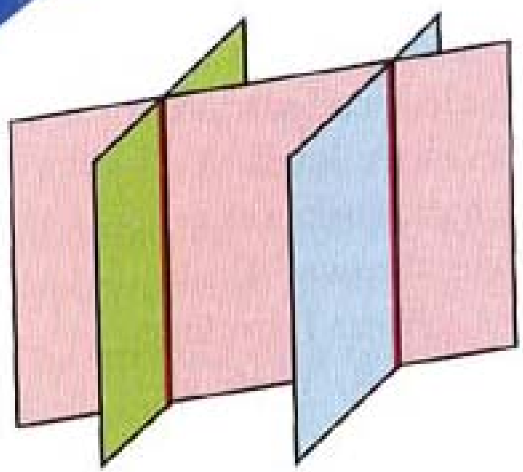
Although we call them linear equations, their graphs are planes in three-dimensional space, not lines.

The solution to a system of three variables and three equations can be shown graphically as the intersections of planes.

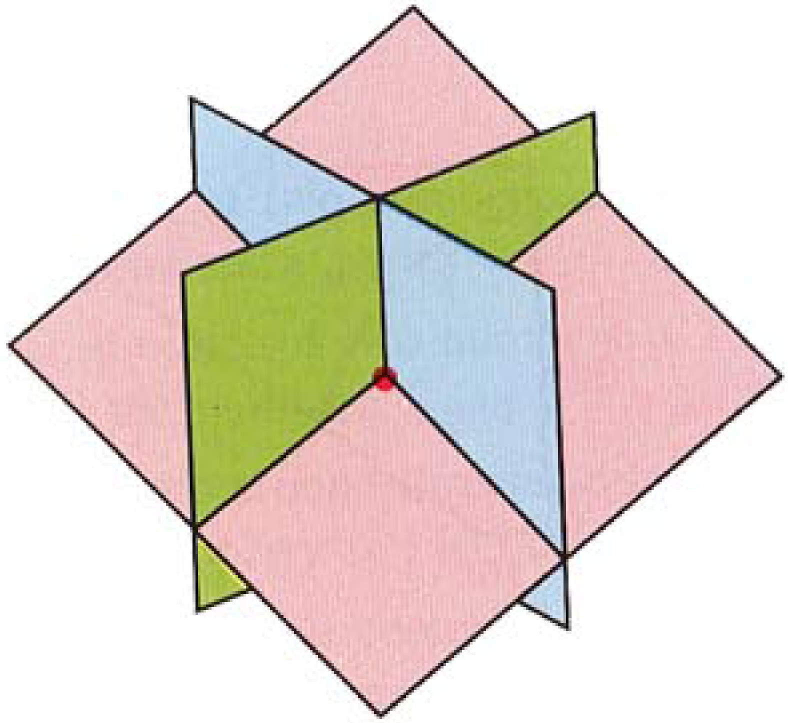
The following systems have no solution because there are no points common to all three planes (equations) and the solution is said to be **inconsistent**.



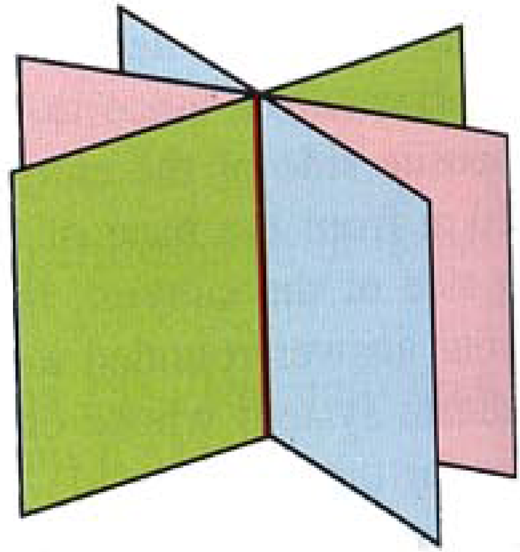




The following system of 3 equations and 3 variables has exactly one solution because all three planes intersect at exactly one point. This system is said to be **consistent**, since there is a solution, and **independent** because the solution is one point (an ordered triple).



The following system has an infinite number of solutions since all three planes intersect to form a line. Since there is a solution, we say the system is **consistent**. Since the solution is a line we also say that the system is **dependent**.



Ex 1) Solve this system:

$$\begin{aligned} 2x + 3y + 2z &= 13 \\ 2y + z &= 1 \\ z &= 3 \end{aligned}$$

Since the third equation tells us that z is equal to 3, let's use the substition method and substitute 3 into the second equation for z, and solve for y.

$$2y + z = 1$$

$$\begin{aligned} 2y + 3 &= 1 \\ -3 \quad &\;\; -3 \end{aligned}$$

$$2y = -2$$

$$\frac{2y}{2} = \frac{-2}{2}$$

$$y = -1$$

Now that we know the values for y and z, let's substitute them both into the first equation and solve for x.

$$2x+3y+2z=13$$

$$2x+3(-1)+2(3)=13$$

$$2x-3+6=13$$

$$2x+3=13$$

$$2x=10$$

$$x=5$$

$(5,-1,3)$ consistent independent

Example 1 illustrates a system that suggests the shape of a triangle. Since triangular systems are easy to solve, this is a good strategy for solving systems of three equations and three variables.

This method is called ***Gaussian Elimination*** in honor of Karl Frederich Gauss who developed this method.

Ex 2) Solve this system:

$$\begin{aligned} x + y - 2z &= 7 \\ -x + 4y + 3z &= 2 \\ 2x - 3y + 2z &= -2 \end{aligned}$$

Let's add the first equation to the second and place the result in the second equation's place. Doing this will eliminate the x variable and leave us with two variables in the second equation's place.

$$\begin{array}{r} x + y - 2z = 7 \\ -x + 4y + 3z = 2 \\ \hline 5y + z = 9 \end{array}$$

$$\begin{aligned} x + y - 2z &= 7 \\ 5y + z &= 9 \\ 2x - 3y + 2z &= -2 \end{aligned}$$

Next, multiply the first equation by -2, and add the result to the third equation and place this result in the third equation's place.

$$-2(x + y - 2z = 7)$$

$$\begin{array}{r} -2x - 2y + 4z = -14 \\ +\ 2x - 3y + 2z = -2 \\ \hline -5y + 6z = -16 \end{array} \rightarrow$$

$$\begin{aligned} x + y - 2z &= 7 \\ 5y + z &= 9 \\ -5y + 6z &= -16 \end{aligned}$$

Now add the second and third equations together to eliminate the y variable and place the result in the third equation's place.

$$\begin{array}{r} 5y + z = 9 \\ -5y + 6z = -16 \\ \hline 7z = -7 \end{array} \longrightarrow \begin{array}{l} x + y - 2z = 7 \\ 5y + z = 9 \\ 7z = -7 \end{array}$$

Let's use the substition method to solve the system.

$$7z = -7$$
$$z = -1 \rightarrow 5y + (-1) = 9$$
$$5y - 1 = 9$$
$$5y = 10$$
$$y = 2$$

$$x + y - 2z = 7$$
$$x + 2 - 2(-1) = 7$$
$$x + 2 + 2 = 7$$
$$x + 4 = 7$$
$$x = 3$$

$(3, 2, -1)$

consistent

independent

Ex 3) Solve the system:

$$\begin{aligned} x - 2y - z &= -1 \\ 2x - y + z &= 1 \\ x + 4y + 5z &= 5 \end{aligned}$$

Multiply the first equation by -2, add it to the second equation and put the result in the second equation's place.

$$-2(x - 2y - z = -1)$$

$$\begin{aligned} -2x + 4y + 2z &= 2 \\ 2x - y + z &= 1 \\ \hline 3y + 3z &= 3 \end{aligned}$$

$$\begin{aligned} x - 2y - z &= -1 \\ 3y + 3z &= 3 \\ x + 4y + 5z &= 5 \end{aligned}$$

Subtract the third equation from the first to eliminate the x variable and place the result in the third equation's place.

$$\begin{aligned} & x - 2y - z = -1 \\ - \; & x + 4y + 5z = 5 \\ \hline & -6y - 6z = -6 \end{aligned}$$

$$\begin{aligned} x - 2y - z &= -1 \\ 3y + 3z &= 3 \\ -6y - 6z &= -6 \end{aligned}$$

Multiply the second equation by 2 and add the result to the third equation and place the result in the third equation's place.

$$2(3y + 3z = 3)$$

$$\begin{aligned} \hookrightarrow \quad 6y + 6z &= 6 \\ + \quad -6y - 6z &= -6 \\ \hline 0 &= 0 \end{aligned}$$

$$\longrightarrow \quad \begin{aligned} x - 2y - z &= -1 \\ 3y + 3z &= 3 \\ 0 &= 0 \end{aligned}$$

Recall that if all the variables of an equation are eliminated, we were instructed to determine whether the result was a **true** statement or a **false** statement.

Since 0 = 0 is a **true** statement, there are an infinite number of solutions to this system of equations.

Therefore, the three planes intersect at one line. Every point on that line is a solution to the system, and we say the system is **consistent** and **dependent**.

If our result was a false statement like 0 = 5, then there would be no solution to the system of equations and we would describe the system as **inconsistent**.